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Bose–Einstein condensation in a gas of the bosonic Newton oscillators

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Abstract

The multi-dimensional q-deformed bosonic Newton oscillator algebra with $U(d)$-symmetry is considered. The high- and low-temperature thermostatistical properties of a gas of the q-deformed bosonic Newton oscillators are obtained in the thermodynamical limit. It is shown that the Bose–Einstein condensation occurs in such a gas for values of the real deformation parameter q smaller than 1. However, the ordinary boson gas results can be recovered in the limit $q = 1$.

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1. Introduction

Quantum groups and quantum algebras are specific deformations of the classical Lie groups and the Lie algebras with some deformation parameter q [1–8]. They have several applications in theoretical physics such as noncommutative geometry [9–11] and exactly solvable statistical models [12, 13].

Furthermore, statistical and thermodynamical consequences of studying q-deformed physical systems have been extensively investigated in the literature [14–43]. Possible connections between quantum groups and Tsallis non-extensive statistical mechanics have been studied [44–46]. In the framework of q-bosons and similar operators called quons [47], some considerable investigations have been carried out for obtaining a possible violation of the Pauli exclusion principle [48] and also a possible relation to anyonic statistics [49–53]. Moreover, several kinds of the one-dimensional q-deformed bosonic and fermionic oscillator algebras have been recently used to study the generalized intermediate statistics [54–60]. However, it was shown in [61–68] that the high- and low-temperature behaviours of 'the quantum group symmetric' bosonic oscillator models depend radically on the real deformation parameters.

In this paper, we consider a different generalization of bosonic system, which is called the multi-dimensional bosonic Newton oscillators. We obtain the high- and low-temperature thermostatistical properties of a gas of the q-deformed bosonic Newton oscillators whose particle algebra is invariant under the undeformed group $U(d)$. In particular, we discuss the effect of the deformation parameter q on the conditions under which the Bose–Einstein condensation would occur.

The paper is organized as follows. In section 2, we review the basic algebraic and representative properties of the multi-dimensional q-deformed bosonic Newton oscillators. In section 3, we investigate the high- and low-temperature behaviours of the q-deformed bosonic Newton oscillator gas with $U(d)$-symmetry. In this context, the distribution function and other thermostatistical functions of the system are derived in terms of two distinct intervals of values for the real deformation parameter q via the grand partition function of the system. In section 4, we discuss the phenomena of Bose–Einstein condensation in the present $U(d)$-invariant q-deformed boson model. In the last section, we give our conclusions.

2. The multi-dimensional bosonic Newton oscillators

In this section, the multi-dimensional q-deformed bosonic Newton oscillator algebra [69–72] invariant under the undeformed group $U(d)$ is presented. Hereafter, the q-deformed bosonic Newton oscillators will be referred to as the BN-oscillators.

The $U(d)$-invariant algebra generated by the BN-oscillators a_i together with their corresponding creation operators a_i^* is defined by the following commutation relations [69]:

$$\begin{aligned} &a_i a_j^* - q a_j^* a_i = q^{\hat{N}} \delta_{ij}, && i, j = 1, 2, \ldots, d, \\ &a_i \hat{N} = (\hat{N}+1) a_i, && a_i a_j - a_j a_i = 0, \end{aligned} \tag{1}$$

where $\hat{N}$ is the total boson number operator in d dimensions, and q is the real positive deformation parameter. The BN-oscillator algebra in equation (1) has the following algebraic and representative properties:

(1) From equation (1), the multi-dimensional undeformed bosonic oscillator algebra can be obtained in the limit $q = 1$.

(2) Under the linear transformation

$$a_i' = \sum_{j=1}^{d} T_{ij} a_j, \tag{2}$$

the relations given in equation (1) are invariant. In this equation, the matrix $T \in U(d)$, and it satisfies the unitarity condition $T\bar{T} = 1$, where the matrix $\bar{T}$is the adjoint matrix of T. This property justifies the name Newton. We note that the same property can be deduced for the fermionic version of the BN-oscillators [73, 74]. We should also mention that the BN-oscillators algebra in equation (1) can be derived from the quantization of the harmonic oscillator through its Newton equation and its invariance properties [69].

(3) The deformed bosonic annihilation operators in equation (1) have the representation [72]

$$a_i = \underbrace{q^{\hat{N}} \otimes q^{\hat{N}} \otimes \ldots \otimes q^{\hat{N}}}_{(i-1)\text{-terms}} \otimes\ a \otimes \underbrace{q^{\hat{N}} \otimes \ldots \otimes q^{\hat{N}}}_{(d-i)\text{-terms}}, \tag{3}$$

where

$$a a^* - q^2 a^* a = q^{2\hat{N}}. \tag{4}$$

From this representation, we also have the following total deformed bosonic number operator for d-dimensional case [69, 71, 72]:

$$\sum_{i=1}^{d} a_i^* a_i = [\hat{N}], \tag{5}$$

whose spectrum is given by

$$[n] = nq^{n-1}, \tag{6}$$

where $n = n_1 + n_2 + \cdots + n_d$. In order to obtain equations (1) and (6), from the above representation, one should consider $q^{1/2}$ instead of q.

(4) The BN-oscillators algebra in equation (1) can alternatively be written [71, 75] as

$$\begin{aligned} &a_i a_j^* - q a_j^* a_i = \tilde{H}\delta_{ij}, \qquad && i, j = 1, 2, \ldots, n, \\ &a_i a_j = a_j a_i, && a_i \tilde{H} = q \tilde{H} a_i, \end{aligned} \tag{7}$$

where the Hermitian operator $\tilde{H}$ can be considered as $q^{\hat{N}}$ after the rescaling of the operators a_i and a_i^*. This algebraic form was recently used to construct a deformed Lie superalgebra $osp_{q_1,q_2}(2n|2m, R)$ in [75].

(5) The quantum group invariant two-parameter deformed bosonic oscillator algebra called the Fibonacci oscillator was recently introduced in [76]. The spectrum of the total deformed bosonic number operator for such oscillators is defined by the generalized Fibonacci basic integer:

$$[n] = \frac{q_1^{2n} - q_2^{2n}}{q_1^2 - q_2^2}, \tag{8}$$

where q_1, q_2 are the real independent deformation parameters. In the limit $q_1 = q_2 = q^{1/2}$, the (q_1, q_2)-deformed boson algebra with $SU_{q_1/q_2}(n)$-symmetry studied in [72, 76] coincides with the present BN-oscillators algebra in equation (1). Therefore, we conclude that the BN-oscillators algebra gives the same spectrum as in the bosonic Fibonacci oscillator algebra in the limit $q_1 = q_2 = q^{1/2}$.

(6) The multi-dimensional two-parameter (q_1, q_2)-oscillators with or without bosonic degeneracy were studied in [77]. It was shown that the limit $q_1 = q_2$ of such two-parameter deformed oscillators coincides and gives the BN-oscillators in equation (1). In this context, the BN-oscillators have bosonic degeneracy for all values of the deformation parameter q.

(7) The one-dimensional case of the BN-oscillators algebra in equation (1) deserves a special attention. The one-dimensional BN-oscillator satisfies the following relations:

$$aa^* - qa^*a = q^{\hat{N}}, \qquad a\hat{N} = (\hat{N} + 1)a, \tag{9}$$

which has another algebraic presentation from equation (7) as

$$aa^* - qa^*a = \tilde{H}, \qquad a\tilde{H} = q\tilde{H}a. \tag{10}$$

The deformed number operator for this oscillator can be calculated as

$$a^*a = \hat{N}q^{\hat{N}-1}, \tag{11}$$

whose spectrum is nq^{n-1}, $n = 0, 1, 2, \ldots$. Thus, one can construct the representations of the operators a, a^* in a Hilbert space spanned on normalized eigenstates $|n\rangle$ of the boson number operator $\hat{N}$:

$$\begin{aligned} &a|n\rangle = \sqrt{nq^{n-1}}|n-1\rangle \qquad \text{with} \qquad a|0\rangle = 0, \\ &a^*|n\rangle = \sqrt{(n+1)q^n}|n+1\rangle. \end{aligned} \tag{12}$$

From the infinite-dimensional matrix representations obtained via this equation, one can calculate the limit $q = 0$. The non-trivial irreducible parts of the representations are

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

One can see from

$$\begin{aligned} &a^2 = 0, && a^{*^2} = 0, \\ &aa^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, && a^*a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{14}$$

that the remarkable commutation relation can be found as

$$aa^* + a^*a = 1 \qquad \text{or} \qquad \{a^*, a\} = 1. \tag{15}$$

Therefore, the $q = 0$ limit of the one-dimensional BN-oscillator algebra gives the one-dimensional fermionic oscillator [71, 78]. However, the multi-dimensional BN-oscillators in equation (1) presents a different generalized bosons with a spectrum given by deformed integer eigenvalues in equation (6).

(8) The one-dimensional case of the BN-oscillators algebra in equation (9) is also different from the following q-bosonic algebra [6–8, 79–81]:

$$\begin{aligned} &cc^* - qc^*c = q^{-N}, \\ &[N, c] = -c, \qquad [N, c^*] = c^*, \\ &[c, c] = [c^*, c^*] = 0. \end{aligned} \tag{16}$$

These operators obey the relations

$$c^*c = [N], \qquad cc^* = [N+1], \tag{17}$$

where the q-basic number is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{18}$$

Recently, this algebra was used to study the thermostatistics of q-deformed bosons obeying the interpolating statistics by Swamy [55, 57]. Furthermore, from the above discussions, the one-dimensional algebra in equation (9) is different from the algebra

$$\begin{aligned} &bb^* - qb^*b = 1, && 0 < q < 1, \\ &[b, N] = b, && [b^*, N] = -b^*, \end{aligned} \tag{19}$$

which was first introduced by Arik and Coon [82]. The number operator spectrum of this bosonic algebra was defined by the relation

$$[n] = \frac{1 - q^n}{1 - q}, \tag{20}$$

which contrasts to the conclusions of the algebra in equations (1) and (6).

All of the properties mentioned above reveal that studying the multi-dimensional BN-oscillators could give new interesting results in the framework of statistical mechanics. These new results may play a role in many different phenomena such as superfluidity, thermodynamic properties of the early universe.

3. Thermostatistical properties of the BN-oscillators

In this section, we investigate the high- and low-temperature thermostatistical properties of the BN-oscillators with $U(d)$-symmetry defined in equations (1) and (6). Similar to the fermionic Newton oscillator gas model studied in [73], the system containing the BN-oscillators constitutes essentially a 'free' q-deformed bosonic gas system, since the BN-oscillators do not interact with each other. The reason behind this consideration is that we do not have both a specific deformed commutation relation between bosonic annihilation (or creation) operators and a quantum group symmetry structure in equation (1). In grand canonical ensemble, we choose the Hamiltonian of such a free q-deformed bosonic Newton oscillator gas as

$$\hat{H}_B = \sum_i (\varepsilon_i - \mu)\hat{N}_i, \tag{21}$$

where ε_i is the kinetic energy of a particle in the state i, μ is the chemical potential which is considered as a function of q, and $\hat{N}_i$ is the boson number operator relative to ε_i. Similar Hamiltonians were also considered by several other researchers [17–25, 27–29, 34–43, 54–59, 73].

Now we derive the q-deformed Bose–Einstein distribution function. First we consider the algebra in equation (1), and then follow the procedure proposed in [83–85]. Taking the thermal averages of both sides in equation (1),

$$\langle a_i a_i^* \rangle + q\langle a_i^* a_i \rangle = \langle q^{\hat{N}} \rangle, \tag{22}$$

leads to the statistical distribution function $[f_{i,q}]$ for the BN-oscillators as

$$[f_{i,q}] \equiv \langle a_i^* a_i \rangle = \frac{\langle q^{\hat{N}} \rangle}{\mathrm{e}^{\beta(\varepsilon_i - \mu)} - q}. \tag{23}$$

By means of equation (6), this can be rewritten as

$$[f_{i,q}] = \frac{q}{\mathrm{e}^{\beta(\varepsilon_i - \mu)} - q}. \tag{24}$$

The same result can also be obtained by employing the principle of detailed balance [19, 20] as follows:

$$\frac{[f_{i,q}]}{[f_{i,q} + 1]} = \exp\{-\beta(\varepsilon_i - \mu)\}, \tag{25}$$

where $[f_{i,q} + 1]$ can be derived from the algebra in equations (1) and (6). We should emphasize that the form of the q-deformed distribution function $[f_{i,q}]$ in equation (24) is different from other studies in the literature [14–43, 54–60], since we used a different realization of the multi-dimensional boson algebra in equation (1). The q-deformed distribution function $[f_{i,q}]$ in equation (24) has the following properties:

(1) In the limit $q = 1$, $[f_{i,q}]$ will be the usual Bose–Einstein distribution.

(2) The distribution function $[f_{i,q}]$ should be nonnegative. This gives the following constraints on the q-deformed fugacity $z_q = \exp(\beta\mu)$ and the chemical potential μ:

$$z_q \leqslant \begin{cases} q^{-1}, & \mu \leqslant -kT\ln q, \quad (q \geqslant 1), \\ q, & \mu \leqslant kT\ln q, \quad (q \leqslant 1). \end{cases} \tag{26}$$

When we take the limit $q = 1$, this equation reduces to

$$z_1 = z \leqslant 1 \qquad \text{or} \qquad \mu = \mu_1 \leqslant 0, \tag{27}$$

as in the case of an undeformed bosonic gas. Also, we note that the q-deformed fugacity z_q is independent of the dimension of the BN-oscillators.

Using the q-deformed statistical distribution function in equation (24), one can find the logarithm of the bosonic grand partition function as

$$\ln Z_B = -\sum_i \ln(1 - qz_q\,\mathrm{e}^{-\beta\varepsilon_i}), \tag{28}$$

which gives all of the thermostatistical functions in terms of the real positive deformation parameter q. Assuming a large volume and particle number, we can replace the summations by integrals. However, we note that the $\vec{p} = \vec{0}$ case plays a special role in the ideal Bose gas [86–88]. Since $\ln Z_B$ diverges in the $\vec{p} = \vec{0}$ term as $z \to 1$, we separately account for the term $\vec{p} = \vec{0}$ as a second term in the following equation of state:

$$\frac{P}{kT} = -\frac{4\pi}{h^3}\int_0^\infty p^2\,\mathrm{d}p\,\ln\left(1 - qz_q\,\mathrm{e}^{-\beta p^2/2m}\right) - \frac{1}{V}\ln(1 - qz_q). \tag{29}$$

Similarly, the particle density for the BN-oscillators is

$$\frac{1}{\upsilon} = \frac{N}{V} = \frac{4\pi}{h^3}\int_0^\infty p^2\,\mathrm{d}p\,\frac{q}{(z_q)^{-1}\,\mathrm{e}^{\beta\,p^2/2m} - q} + \frac{1}{V}\frac{q}{(z_q)^{-1} - q}. \tag{30}$$

Equations (29) and (30) can be rewritten as

$$\frac{P}{kT} = \frac{1}{\lambda^3}g_{5/2}(q, z_q) - \frac{1}{V}\ln(1 - qz_q), \tag{31}$$

$$\frac{1}{\upsilon} = \frac{1}{\lambda^3}g_{3/2}(q, z_q) + \frac{1}{V}\frac{q}{(z_q)^{-1} - q}, \tag{32}$$

where $\lambda = \sqrt{2\pi\hbar^2/mkT}$ is the thermal wavelength. The generalized Bose–Einstein functions $g_{5/2}(q, z_q)$ and $g_{3/2}(q, z_q)$ are defined as follows:

$$g_{5/2}(q, z_q) = -\frac{4}{\sqrt{\pi}}\int_0^\infty x^2\,\mathrm{d}x\,\ln(1 - qz_q\,\mathrm{e}^{-x^2}) = \sum_{l=1}^\infty \frac{(qz_q)^l}{l^{5/2}}, \tag{33}$$

$$g_{3/2}(q, z_q) = \frac{4}{\sqrt{\pi}}\int_0^\infty \frac{x^2\,\mathrm{d}x}{(qz_q)^{-1}\,\mathrm{e}^{x^2} - 1} = \sum_{l=1}^\infty \frac{(qz_q)^l}{l^{3/2}}, \tag{34}$$

where $x^2 = \beta p^2/2m$. These generalized functions reduce to the standard Bose–Einstein functions $g_n(z)$ in the limit $q = 1$. They are also different from $h_n(z, q)$ in [55].

In figures 1 and 2, the q-deformed functions $g_{3/2}(q, z_q)$ and $g_{5/2}(q, z_q)$ are shown as a function of z for different values of the deformation parameter q, respectively. As shown in equation (26), in these figures, the upper bound of z is $1/q$ for $q \geqslant 1$, and q for $q \leqslant 1$. When we compare with the $q = 1$ case in figures 1 and 2, the values of the q-deformed Bose–Einstein functions $g_{3/2}(q, z_q)$ and $g_{5/2}(q, z_q)$ decrease for $q < 1$, while they increase for $q > 1$.

Equations (24) and (32) imply that $\langle n_0\rangle$ is the average occupation number for the zero momentum state,

$$\langle n_0\rangle = \frac{qz_q}{1 - qz_q}. \tag{35}$$

This term contributes significantly to equation (32) if $\langle n_0\rangle/V$ is a finite number, i.e., if a finite fraction of the BN-oscillators occupies the single level with $\vec{p} = \vec{0}$. This fact gives rise to the famous phenomenon of Bose–Einstein condensation. In this context, one can rewrite equation (32) as

$$\frac{\lambda^3\,\langle n_0\rangle}{V} = \frac{\lambda^3}{\upsilon} - g_{3/2}(q, z_q), \tag{36}$$

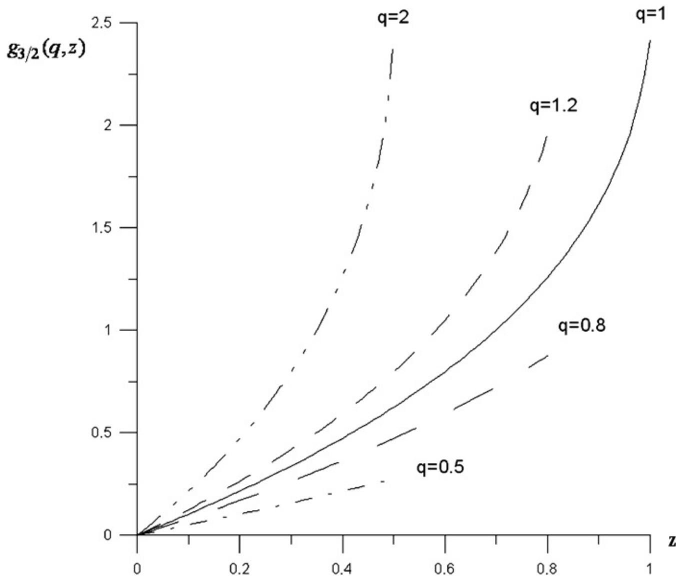

Figure 1. The q-deformed Bose–Einstein function $g_{3/2}(q, z)$ as a function of z for different values of the deformation parameter q.

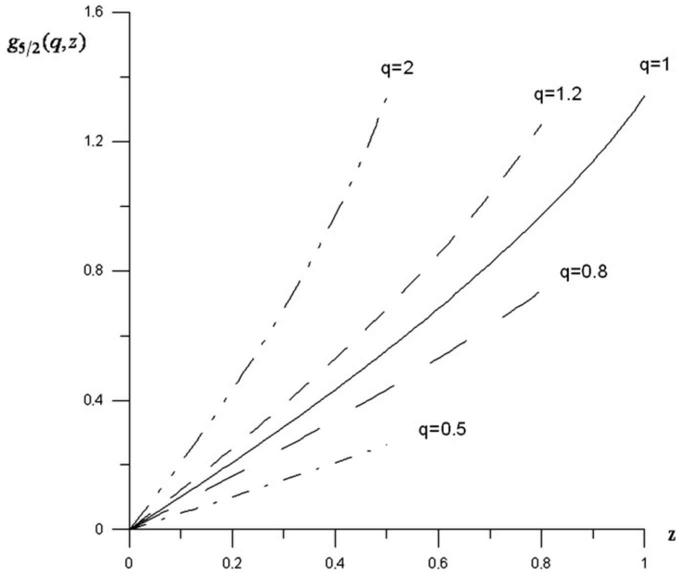

Figure 2. The q-deformed Bose–Einstein function $g_{5/2}(q, z)$ as a function of z for different values of the deformation parameter q.

which implies $(\langle n_0 \rangle / V) > 0$, when the critical combination of the temperature and the specific volume occurs such that the q-deformed fugacity z_q will reach its maximum value given in equation (26). Therefore, we obtain

$$\frac{\lambda^3}{\upsilon} \geqslant g_{3/2}(q, z_q). \tag{37}$$

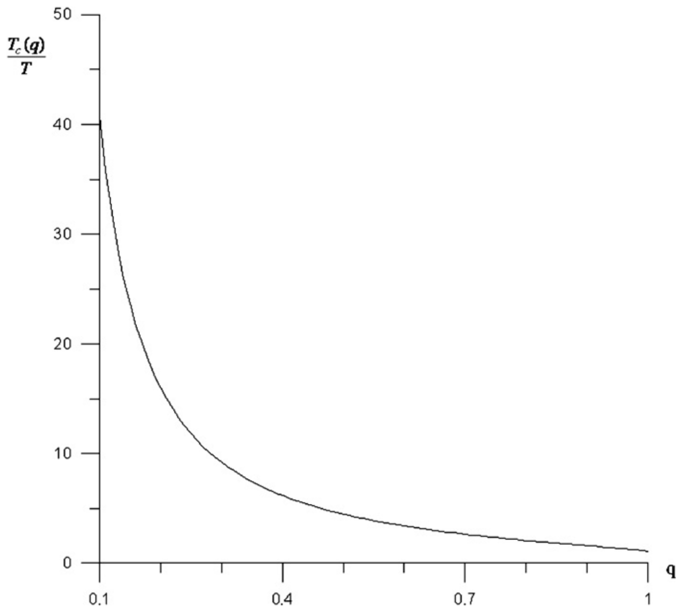

Figure 3. The ratio $T_c(q)/T_c(1)$ of the q-deformed critical temperature $T_c(q)$ and the undeformed $T_c(1)$ as a function of the deformation parameter q.

This phenomenon is referred to as the Bose–Einstein condensation. In the framework of the constraints in equation (26), we consider the case $q \leqslant 1$ in the rest of the calculations of this study. Since the function $g_{3/2}(q, z_q)$ with $q \geqslant 1$ in equation (34) gives the same results as in the case of an undeformed boson gas. Hence, the low-temperature behaviour of the present BN-oscillators model is interesting only for values of the deformation parameter q smaller than 1.

The critical temperature $T_c(q)$ for the BN-oscillators gas can be found from equation (37) as

$$T_c(q) = \frac{2\pi\hbar^2/mk}{[\upsilon g_{3/2}(q, z_q)]^{2/3}}. \tag{38}$$

Thus, from figure 1, the critical temperature for the BN-oscillators is much larger than the critical temperature $T_c(1)$ for an undeformed boson gas in the special region of the deformation parameter q close to zero. Obviously, one can find a relation between the critical temperature of the present BN-oscillators gas and of the undeformed boson gas:

$$\frac{T_c(q)}{T_c(1)} = \left(\frac{2.61}{g_{3/2}(q, z_q)}\right)^{2/3}. \tag{39}$$

In figure 3, we show the plot of equation (39) as a function of the deformation parameter q for the case $q \leqslant 1$.

The internal energy U of the BN-oscillators gas can be found by $U = (-\partial \ln Z_B/\partial\beta)$, which leads to

$$\frac{U}{V} = \frac{3}{2}\frac{kT}{\lambda^3} g_{5/2}(q, z_q). \tag{40}$$

With the above results in mind, the specific heat of the BN-oscillators gas can be obtained from $C_V = (\partial U/\partial T)_V$. For low temperatures, namely in the limit $T < T_c(q)$, the specific

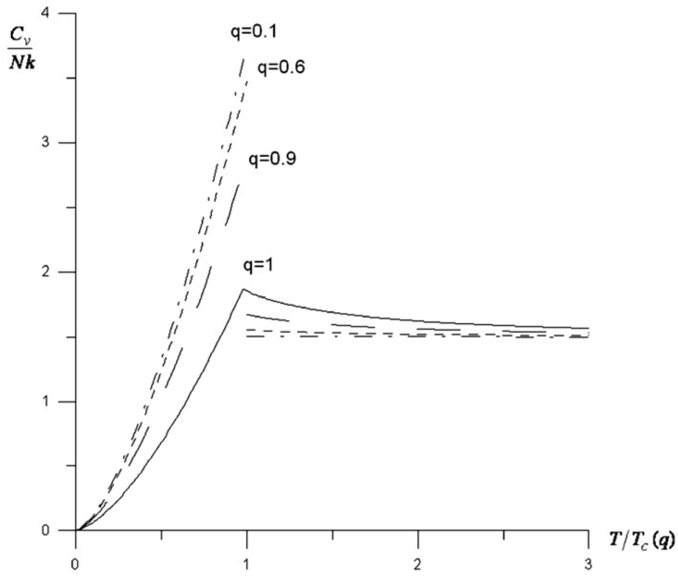

Figure 4. The specific heat C_V/Nk as a function of $T/T_c(q)$ for values of the deformation parameter q smaller than 1.

heat of our model is

$$\frac{C_V}{Nk} = \frac{15}{4}\frac{\upsilon}{\lambda^3} g_{5/2}(q, z_q), \tag{41}$$

which can be rewritten in terms of the critical temperature $T_c(q)$ by means of equation (38) as

$$\frac{C_V}{Nk} = \frac{15}{4}\frac{g_{5/2}(q, z_q)}{g_{3/2}(q, z_q)}\left(\frac{T}{T_c(q)}\right)^{3/2}. \tag{42}$$

On the other hand, we have the following specific heat for the BN-oscillators gas in the limit $T > T_c(q)$:

$$\frac{C_V}{Nk} = \frac{15}{4}\frac{\upsilon}{\lambda^3} g_{5/2}(q, z_q) - \frac{9}{4}\frac{g_{3/2}(q, z_q)}{g_{1/2}(q, z_q)}, \tag{43}$$

which can be approximated as

$$\frac{C_V}{Nk} \approx \frac{3}{2} + \frac{3}{4.2^{5/2}} g_{3/2}(q, z_q)\left(\frac{T_c(q)}{T}\right)^{3/2}. \tag{44}$$

From equations (42) and (44), we deduce the gap in the specific heat in the limit $T = T_c(q)$:

$$\frac{\Delta C_V}{Nk} \approx \left\{\frac{15}{4}\frac{g_{5/2}(q, z_q)}{g_{3/2}(q, z_q)} - \left[\frac{3}{2} + \frac{3}{4.2^{5/2}} g_{3/2}(q, z_q)\right]\right\}. \tag{45}$$

In figure 4, we show the plot of the specific heat C_V/Nk as a function of $T/T_c(q)$ for values of the deformation parameter q smaller than 1. In figure 5, we also show the plot of the gap in specific heat $\Delta C_V/Nk$, using equation (45), as a function of the deformation parameter q for the case $q \leqslant 1$.

On the other hand, the entropy for the BN-oscillators gas can be obtained from $S = (U - F)/T$. For low temperatures, the entropy for our model is

$$\frac{S}{Nk} = \frac{5}{2}\frac{\upsilon}{\lambda^3} g_{5/2}(q, z_q), \tag{46}$$

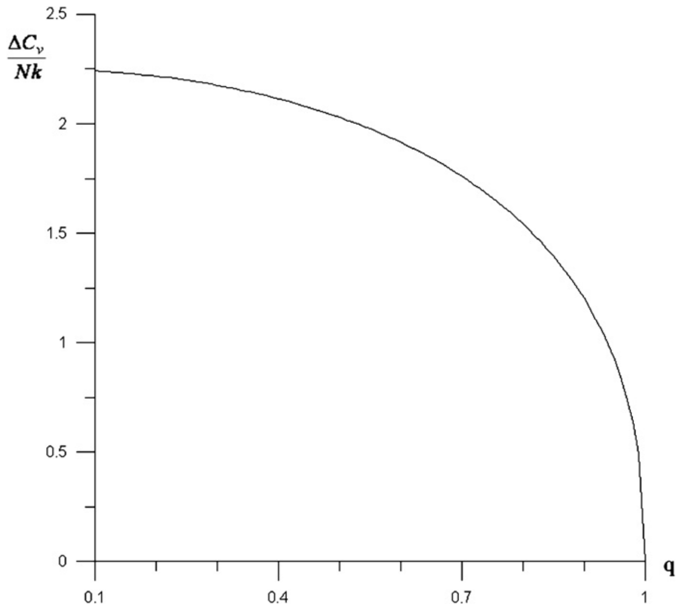

Figure 5. The gap in the specific heat $\Delta C_V/Nk$ at the critical temperature $T_c(q)$ as a function of the deformation parameter q.

and similarly, the entropy for high temperatures is

$$\frac{S}{Nk} = \frac{5}{2}\frac{\upsilon}{\lambda^3} g_{5/2}(q, z_q) - \ln z_q. \tag{47}$$

From equations (46), (47) and (26), the jump of the entropy at $T = T_c(q)$ can be obtained as

$$\frac{\Delta S}{V} = \frac{k}{\lambda^3} g_{3/2}(q, z_q) \ln q, \tag{48}$$

where the case $q < 1$ is considered. The entropy of the BN-oscillators gas gives the same results as the entropy of an undeformed boson gas in the limit $q = 1$ [86–88]. We observe from equation (48) that for $q < 1$, the entropy values of the BN-oscillators gas at the critical point is different than those of an undeformed boson gas. The jump of the entropy of the BN-oscillators gas at the critical point increases with the values of the deformation parameter q up to a value $q = 0.67$.

By considering the above results, the effect of the deformation parameter q on the thermostatistics of the BN-oscillators gas will be discussed in the following section.

4. Discussion

As shown in figure 4, the specific heat of the BN-oscillators gas shows a discontinuity at the critical temperature. This means that the Bose–Einstein condensation in the BN-oscillators gas is a second-order phase transition. Also, the specific heat of the BN-oscillators gas has a λ-point transition behaviour which is not exhibited by an undeformed boson gas. This could provide some implications in studies on superconductivity or superfluidity. An interesting point is that when the deformation parameter q approaches zero, the discontinuity in the specific heat of the system increases (figure 4). Conversely, it disappears in the limit $q = 1$, showing an undeformed boson gas behaviour.

Furthermore, the gap in the specific heat of the BN-oscillators gas at the condensation temperature decreases with the values of the deformation parameter q up to the limit $q = 1$ (figure 5). Although, many studies in the literature [14–43] are devoted to the region $q \geqslant 1$, our study is an example of a deformed boson system with a deformation parameter smaller than 1. It shows the Bose–Einstein condensation for low temperatures in the interval $0 < q < 1$.

Also, the theoretical critical temperature for ^{4}He, $T_c \approx 3.13$ K [86–88], corresponds to a q value of about 0.9. It is interesting to note that the same value $q \approx 0.9$ fits very well the gap in the specific heat of a dilute gas of rubidium atoms [89]. Such a result may be physically important, since some recent studies similarly adduced a value of the deformation parameter q. Hence, one might well view q-deformation as a phenomenological means of introducing an extra parameter, 'q', to account for nonlinearity in the system. Such an approach was considered in [90], where a value of q is found to fit the properties of a real (non-ideal) laser.

As a final discussion, we emphasize that the multi-dimensional BN-oscillator algebra [69] is a newly developed algebraic structure, since it has not been fully examined in the past in the literature. Moreover, we wish to outline some structural property of the algebra of the BN-oscillators together with its fermionic version studied in [73]. We could consider the bosonic and fermionic Newton oscillator algebras with $U(d)$-symmetry as a different manifestation of the same structure:

$$\begin{aligned} &a_i a_j^* - \kappa q a_j^* a_i = q^{\hat{N}} \delta_{ij}, \qquad i, j = 1, 2, \ldots, d, \\ &a_i a_j - \kappa a_j a_i = 0, \end{aligned} \tag{49}$$

where κ is used to describe boson-like particles for $\kappa = +1$ and fermion-like particles for $\kappa = -1$, respectively. Obviously, the undeformed bosonic and fermionic oscillator algebras can be obtained in the limit $q = 1$. Even more interesting is the fact that the q-deformed statistical distribution functions for both the present BN- and the FN-oscillator [73] gases can be reconsidered as

$$[f_{i,q}] = \frac{q}{\mathrm{e}^{\beta(\varepsilon_i - \mu)} - \kappa q}. \tag{50}$$

In the following section, we will give some concluding remarks about the above discussions.

5. Conclusions

In this paper, we studied the algebraic and representative properties of the BN-oscillators. The Hamiltonian of this system does not show invariance under a quantum group structure. Hence, the system of the BN-oscillators constitutes essentially an example of non-interacting multi-mode system of the q-deformed bosonic particles. The algebra of the BN-oscillators has the following properties: It has $U(d)$-symmetry, and the deformation parameter q can have values in the interval $0 < q < \infty$.

Furthermore, we discussed the low- and high-temperature behaviours in a gas of the BN-oscillators. Starting with a q-deformed Bose–Einstein distribution function, several thermostatistical functions via the grand partition function of the system are calculated. Due to the algebraic reasons originating from equations (24), (26), (33), and (34), we obtained such thermostatistical functions in terms of the deformation parameter q for its specific interval $0 < q \leqslant 1$. For instance, the average occupation number, the critical temperature, the entropy are derived for low temperatures. Subsequently, the specific heat of the system is obtained in the low- and high-temperature limits. We then focused on the effect of the deformation parameter q on these results. We should emphasize that the values of these deformed functions and thus all other thermostatistical functions will be more sensitive to those q values, which are small deviations of the deformation parameter q close to zero. However, the results for an undeformed boson gas can be recovered in the limit $q = 1$.

As a final remark, our studies reveal that the bosonic and fermionic Newton oscillator algebras in equations (49) and (50) may serve as a new candidate to study systems with fractional statistics such as anyon-like particles. A parallel discussion is on possible consequences of the bosonic and fermionic Newton oscillator algebras with the deformation parameter q being a root of unity. We hope that detailed discussions on such problems will be reported in future publications.

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